Occupation time fluctuations of Poisson and equilibrium finite variance branching systems

Piotr Miłoś
Institute of Mathematics
Polish Academy of Sciences
Warsaw

2nd February 2008

Abstract

Functional limit theorems are presented for the rescaled occupation time fluctuation process of a critical finite variance branching particle system in $\mathbb{R}^d$ with symmetric α-stable motion starting off from either a standard Poisson random field or from the equilibrium distribution for intermediate dimensions $\alpha < d < 2\alpha$. The limit processes are determined by sub-fractional and fractional Brownian motions, respectively.

AMS subject classification: primary 60F17, 60G20, secondary 60G15

Key words: Functional central limit theorem; Occupation time fluctuations; Branching particles systems; Fractional Brownian motion; Sub-fractional Brownian motion; equilibrium distribution.

1 Introduction

Consider a system of particles in $\mathbb{R}^d$ starting off at time $t = 0$ from a certain distribution (a standard Poisson and equilibrium fields are investigated in this paper). They evolve independently, moving according to a symmetric α-stable Lévy process and undergoing finite variance branching at rate V $(V > 0)$. We obtain functional limit theorems for the rescaled occupation time fluctuations of this system when $\alpha < d < 2\alpha$. This is an extension of [4, Theorem 2] where the starting distribution is a Poisson field and the branching law is critical and binary.

1.1 Branching law

In the [3, 4, 5] the law of branching is critical and binary. In this paper an

extended model is investigated. The particles branch according to the law given by a moment generating function F. F fulfills two requirements:

1. $F'(1) = 1$, which means that the law is critical (the expected number of particles spawning from one particle is 1),
2. $F''(1) < +\infty$, which states that the second moment exists.

(Note here that the branching law in [4] is given by $F(s) = \frac{1}{2}\left(1+s^2\right)$ and obviously fulfills the two requirements.) Although constraints imposed on F are not very restrictive and quite natural (so that the class of the branching laws satisfying them is broad) still there remain other interesting cases to be investigated. One of them is the class of branching laws in the domain of attraction of the $(1+\beta)$ $-$stable law (i.e., the moment generating function is $F(s) = s + \frac{1}{2}(1+s)^{1+\beta}$), the case studied in [6, 7]. A remarkable feature of the latter case is that the limit processes are stable ones and not Gaussian as it occurs in the finite variance case.

1.2 Equilibrium distribution

Another concept naturally related to particle systems is an equilibrium distribution. It has been shown that in certain circumstances the system converges to the equilibrium distribution [12]. It is both an interesting and important question whether the theorems shown by Bojdecki et al still hold in the case when the equilibrium state is taken as the initial condition. A conjecture in [3] states that the temporal structure of the limit is given by fractional Brownian motion. It is of interest to notice that the limit is different from the one in the case of the system starting off from the Poisson field (where temporal structure is sub-fractional Brownian motion). We study behavior of the system for a branching law given by F. But there is still broad area for further studies. No attempt has been made to develop more general theory concerning systems with a general starting distribution (or a large class of distributions).

1.3 General concepts and notation

Let us denote N_t^{Poiss} and N_t^{eq}, the empirical processes for the system starting off from the Poisson field with Lebesgue intensity measure and the equilibrium respectively. For a measurable set $A \subset \mathbb{R}^d$, $N_t^{Poiss}(A)$, $N_t^{eq}(A)$, respectively are the numbers of particles of the system in set A at time t. Note that they are measure-valued processes but we will consider them as processes with values in $\mathcal{S}'$ (the space of tempered distributions) because this space has good analytical properties.

The equilibrium distribution is defined by

$$\lim_{t\to+\infty} N_t^{Poiss} = N_{eq},$$

where the limit is understood in weak sense. The Laplace functional of the equilibrium distribution is given by

$$\mathbb{E}\exp\{-\langle N_{eq},\varphi\rangle\} = \exp\left\{\left\langle \lambda, e^{-\varphi}-1\right\rangle + V\int_0^\infty \langle\lambda, H(j(\cdot,s))\rangle\, ds\right\}, \quad (1.1)$$

where

$$j(x,l) := \mathbb{E}\exp\left(-\langle N_l^x,\varphi\rangle\right) \quad (1.2)$$

$H(s) = F(s) - s$, $\varphi : \mathbb{R}^d \to \mathbb{R}_+$, $\varphi \in \mathcal{L}^1(\mathbb{R}^d)\cap C(\mathcal{R}^d)$ and j satisfies the integral equation

$$j(x,l) = \mathcal{T}_l e^{-\varphi}(x) + V\int_0^l \mathcal{T}_{l-s} H(j(\cdot,s))(x)\, ds,$$

This equations can be obtained in the same way as [12, (2.4)]. Note that in [12] function φ is continuous with compact support. We approximate $\varphi \in \mathcal{L}^1$ using functions φ_n with compact support $\varphi_n \nearrow \varphi$. Using Lebesgue's monotone convergence theorem it is easy to obtain the above equations for φ (H is decreasing because of the criticality of the branching law).

For an empirical process N_t the rescaled occupation time fluctuation process is defined by

$$X_T(t) = \frac{1}{F_T}\int_0^{Tt} (N_s - \mathbb{E}N_s)\, ds,\ t \geq 0, \quad (1.3)$$

where $T > 0$ and F_T is a suitable norming. We are interested in the weak functional limit of X_T when time is accelerated (i.e., T tends to ∞).

The α-stable process starting from x will be denoted by η_t^x its semigroup by $\mathcal{T}_t$ and its infinitesimal operator by Δ_α. The Fourier transform of $\mathcal{T}_t$ is

$$\widehat{\mathcal{T}_t\varphi}(z) = e^{-t|z|^\alpha}\widehat{\varphi}(z). \quad (1.4)$$

For brevity let us denote

$$K = \frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)\, h\,(h-1)}, \quad (1.5)$$

where

$$h = 3 - d/\alpha \quad (1.6)$$

(in this paper we always assume that $\alpha < d < 2\alpha$ so $h > 1$) and

$$M = F''(1). \quad (1.7)$$

We will now introduce two centered Gaussian processes. One of them is sub-fractional Brownian motion with parameter h with the covariance function C_h

$$C_h(s,t) = s^h + t^h - \frac{1}{2}\left[(s+t)^h + |s-t|^h\right] \quad (1.8)$$

and the second one is fractional Brownian motion with parameter h and the covariance function c_h

$$c_h(s,t) = \frac{1}{2}\left(s^h + t^h - |s-t|^h\right). \quad (1.9)$$

1.4 Space-time method

The space-time method is a very convenient technique for investigating the weak convergence in the $C\left([0,\tau],\mathcal{S}'\left(\mathbb{R}^d\right)\right)$ space. It was developed by Bojdecki et al and can be found in [2]. If $X=(X(t))_{t\in[0,\tau]}$ is a continuous $\mathcal{S}'\left(\mathbb{R}^d\right)$-valued process we define a random element $\tilde{X}$ of $\mathcal{S}'\left(\mathbb{R}^{d+1}\right)$ by

$$\left\langle \tilde{X},\Phi\right\rangle = \int_0^\tau \left\langle X(t),\Phi(\cdot,t)\right\rangle dt, \tag{1.10}$$

where $\Phi\in\mathcal{S}\left(\mathbb{R}^{d+1}\right)$. In order to prove that X_T converges weakly to X in $C\left([0,\tau],\mathcal{S}'\left(\mathbb{R}^d\right)\right)$ it suffices to show that

$$\left\langle \tilde{X}_T,\Phi\right\rangle \Rightarrow \left\langle \tilde{X},\Phi\right\rangle,\ \forall_{\Phi\in\mathcal{S}(\mathbb{R}^{d+1})}$$

and that the family X_T is tight.

2 Convergence theorems

We will present two theorems. In the first of them (which is a direct extension of [4, Theorem 2.2]) we study the occupation time fluctuation process for the branching system starting off from the Poisson field with Lebesgue intensity measure (denoted by λ) with the branching law given by a moment generating function as described in Section 1.1. The result is very similar to the one obtained in [4, Theorem 2.2] - namely, the limit process is the same up to constants.

Theorem 2.1. *Assume that $\alpha<d<2\alpha$ and let X_T be the occupation time fluctuation process defined by (1.3) for the branching system N^{Poiss}, and $F_T=T^{\left(3-\frac{d}{\alpha}\right)/2}$. Then $X_T\Rightarrow X$ in $C\left([0,\tau],\mathcal{S}'\left(\mathbb{R}^d\right)\right)$ as $T\to+\infty$ for any $\tau>0$, where $(X(t))_{t\geq 0}$ is a centered $\mathcal{S}'$-valued, Gaussian process with covariance function:*

$$Cov\left(\langle X(s),\varphi\rangle,\langle X(t),\psi\rangle\right)=KM\langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle C_h(s,t), \tag{2.1}$$

where $\varphi,\psi\in\mathcal{S}\left(\mathbb{R}^d\right)$.

The second theorem concerns the case where the system starts from the equilibrium distribution. As it was mentioned hereinabove the theorem is interesting because the limit has a different time structure from the one in [4, Theorem 2.2] and Theorem 2.1.

Theorem 2.2. *Assume that $\alpha<d<2\alpha$ and let X_T be the occupation time fluctuation process defined by (1.3) for the branching system N^{eq}, and $F_T=T^{\left(3-\frac{d}{\alpha}\right)/2}$. Then $X_T\Rightarrow X$ in $C\left([0,\tau],\mathcal{S}'\left(\mathbb{R}^d\right)\right)$ as $T\to+\infty$ for any $\tau>0$, where $(X(t))_{t\geq 0}$ is a centered Gaussian process with the covariance function*

$$Cov\left(\langle X(s),\varphi\rangle,\langle X(t),\psi\rangle\right)=KM\langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle c_h(s,t), \tag{2.2}$$

where $\varphi, \psi \in \mathcal{S}\left(\mathbb{R}^d\right)$.

Remark 2.3. The limit processes above can be represented as follows:
For Theorem 2.1

$$X = (MK)^{1/2}\, \lambda\beta^h$$

and for Theorem 2.2

$$X = (MK)^{1/2}\, \lambda\xi^h,$$

where β^h and ξ^h are respectively sub-fractional and fractional Gaussian processes defined in Section 1.3. In both cases the limit process X has a trivial spatial structure (Lebesgue measure), whereas the time structure is complicated, with long range dependence.

Remark 2.4. The occupation time fluctuation processes of particle systems form an area that receives a lot of research attention. We would like to mention some other related work. Firstly the case of non-branching systems has been studied in [4, Theorem 2.1]. The result is analogous, both to Theorem 2.1 and 2.2 because the Poisson field is the equilibrium distribution for the system. The limit process is essentially the same as in Theorem 2.2. For the critical $d = 2\alpha$ and large dimensions $d > 2\alpha$, there is no long range dependence and the results can be found in [5]. In [8] the fluctuations of the occupation time of the origin are studied for a critical binary branching random walks on the d-dimensional lattice, $d \geq 3$, including also the equilibrium case. The convergence results are analogous to those in [4, 5] and in this paper but the proofs are substantially different. A similar model with $\alpha = 2$ was investigated in [9] (ie. with particles moving according to Brownian motion).

3 Proofs

The main idea used in both of the proofs is to study the Laplace functional of a process given by the space-time method. The Fourier transform is used for this purpose. This is similar to the method in [4]. In the case of Theorem 2.1 the proof follows the same principle as [4, Theorem 2.2]. The moment generating function can be represented using Taylor's expansion and two following statements need to be proved. Firstly, one has to check that the method used in [4] can still be applied. Secondly, it needs to be shown that terms of order higher then 2 play no role in the limit. The proof of Theorem 2.2 requires more work. The Laplace formula contains a function that is a solution of a differential equation. This makes the computations more cumbersome. Some expressions in this proof had to be examined more carefully than in Theorem 2.1. It should be noted that Theorem 2.2 covers all branching laws described in Section 1.1. Now we introduce some notation and facts used further on.

For a generating function F we define

$$G(s) = F(1-s) - 1 + s. \tag{3.1}$$

The following fact describes basic properties of G which are straightforward consequences of the properties of F.

Fact 3.1. *1.* $G(0) = F(1) - 1 = 0$,

2. $G'(0) = -F'(1) + 1 = 0$ *since* $F'(1) = 1$,

3. $G''(0) = F''(1) < +\infty$,

4. $G(v) = \frac{M}{2} v^2 + g(v) v^2$ *where* M *is defined by (1.7) and* $\lim_{v \to 0} g(v) = 0$.

The next simple fact will be useful in proving some inequalities

Fact 3.2. $G(v) \geq 0$ *for* $v \in [0,1]$.

Proof. $F''(1-v) \geq 0$ which is an obvious consequence of the fact that all of the coefficients in the expansion of F'' are non-negative and $1 - v \in [0,1]$. $G''(v) = F''(1-v) \geq 0$. We also know that $G'(0) = 0$ so $G'(v) \geq 0$ for $v \in [0,1]$. The proof is complete since $G(0) = 0$ and G is non-decreasing. □

The existence of the second moment of the moment generating function F implies also that G is comparable with function v^2.

Fact 3.3. *We have*

$$\sup_{v \in [0,1]} \frac{G(v)}{v^2} < +\infty$$

Proof. Since both $G(v)$ and v^2 are continuous we only have to check that the limit of the quotient at $v = 0$ is finite. This becomes obvious when we recall Taylor's expansion of $G(v)$ from Fact 3.1, property 4. □

Let us now introduce some notation used throughout the rest of the paper. Φ will denote a positive function from $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$. [4, Lemma in Section 3.2] explains why without loss of generality it can be assumed $\Phi \geq 0$. We denote

$$\Psi(x,s) = \int_s^1 \Phi(x,t)\, dt,$$

$$\Psi_T(x,s) = \frac{1}{F_T} \Psi\left(x, \frac{s}{T}\right).$$

To make computations less cumbersome we will sometimes assume that Φ is of the form $\Phi(x,t) = \varphi(x)\psi(t)$ for $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, $\psi \in \mathcal{S}(\mathbb{R})$ and hence

$$\Psi_T(x,t) = \varphi_T(x)\chi_T(t), \tag{3.2}$$

where $\varphi_T(x) = \frac{1}{F_T}\varphi(x)$, $\chi(t) = \int_t^1 \psi(s)\, ds$, $\chi_T = \chi\left(\frac{t}{T}\right)$. Notice that $\varphi \geq 0, \chi \geq 0$ as $\Phi \geq 0$.
Let us introduce now an important function which will appear as a part of the Laplace functional of the occupation time fluctuation processes

$$v_\Psi(x,r,t) = 1 - \mathbb{E}\exp\left\{-\int_0^t \langle N_s^x, \Psi(\cdot, r+s)\rangle\, ds\right\},$$

where N_s^x denotes the empirical measure of the particle system with the initial condition $N_0^x = \delta_x$. Let us note here that due to the fact that $\Psi \geq 0$ we have $v_\Psi \in [0,1]$. We also write

$$n_\Psi(x,r,t) = \int_0^t \mathcal{T}_{t-s}\Psi(\cdot, r+t-s)(x)\, ds. \tag{3.3}$$

For simplicity of notation, we write

$$v_T(x,r,t) = v_{\Psi_T}(x,r,t), \tag{3.4}$$

$$n_T(x,r,t) = n_{\Psi_T}(x,r,t), \tag{3.5}$$

$$v_T(x) = v_T(x,0,T), \tag{3.6}$$

$$n_T(x) = n_T(x,0,T) \tag{3.7}$$

when no confusion can arise.

Now we obtain an integral equation for v which will play a crucial role in the next proofs. Note that similar computations can be found also in [11].

Lemma 3.4. *v_Ψ satisfies the equation*

$$v_\Psi(x,r,t) = \int_0^t \mathcal{T}_{t-s}\left[\Psi(\cdot, r+t-s)(1 - v_\Psi(\cdot, r+t-s, s)) - VG(v_\Psi(x, r+t-s, s))\right](x)\, ds. \tag{3.8}$$

Proof. Firstly let us investigate

$$w(x,r,t) \equiv w_\Psi(x,r,t) = \mathbb{E}\exp\left(-\int_0^t \langle N_s^x, \Psi(\cdot, r+s)\rangle\, ds\right) = 1 - v_\Psi(x,r,t),$$

We assume $\Psi \geq 0$ hence we have $w(x,r,t) \in [0,1]$. By conditioning on the time of the first branching we obtain the following equation

$$\begin{aligned} w(x,r,t) =& e^{-Vt}\mathbb{E}\left(-\int_0^t \Psi(\eta_s^x, r+s) ds\right) \\ &+ V\int_0^t e^{-Vs}\mathbb{E}\exp\left(-\int_0^s \Psi(\eta_u^x, r+u) du\right) F\left(w\left(\eta_s^x, r+s, t-s\right)\right), \end{aligned}$$

where $t \geq 0$, $r \geq 0$.
Using Feynman-Kac formula one can obtain the following equation for w (for details see [4, (3.13)-(3.17)])

$$\begin{cases} \frac{\partial}{\partial t} w(x,r,t) = \left(\Delta_\alpha + \frac{\partial}{\partial r} - \Psi(x,r)\right) w(x,r,t) + V\left[F(w(x,r,t)) - w(x,r,t)\right], \\ w(x,r,0) = 1. \end{cases}$$

$v(x,r,t) = v_\Psi(x,r,t) = 1 - w_\Psi(x,r,t)$ so v satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t} v(x,r,t) = \left(\Delta_\alpha + \frac{\partial}{\partial r}\right) v(x,r,t) + \Psi(x,r)(1 - v(x,r,t)) - VG(v(x,r,t)), \\ v(x,r,0) = 0. \end{cases}$$

Its integral version is 3.8 (note that in [4] $G(t) = \frac{1}{2}t^2$).

$$v(x,r,t) = \int_0^t \mathcal{T}_{t-s}\left[\Psi(\cdot, r+t-s)(1 - v(\cdot, r+t-s,s)) - VG(v(x,r+t-s,t))\right](x)\,ds.$$

□

Fact 3.5.

$$v_\Psi(x,r,t) \le n_\Psi(x,r,t) \tag{3.9}$$

Proof. This is a direct consequence of the equation (3.8), the fact that $1 \ge v \ge 0$ and Fact 3.2. □

Fact 3.6. *For the system N_t^{Poiss} the covariance function is given by*

$$Cov\left(\left\langle N_u^{Poiss}, \varphi\right\rangle, \left\langle N_v^{Poiss}, \psi\right\rangle\right) = \langle \lambda, \varphi \mathcal{T}_{v-u}\psi\rangle \tag{3.10}$$
$$F''(1) \cdot V \int_0^u \langle \lambda, \varphi \mathcal{T}_{u+v-2r}\psi\rangle\,dr, \quad u \le v,$$

where $\varphi, \psi \in \mathcal{S}\left(\mathbb{R}^d\right)$.

The proof of the fact follows from a simple computation which can be carried on using [10, formula (3.14)], therefore we omit it.

3.1 Proof of theorem 2.1

3.1.1 Tightness

The first step required to establish the weak convergence is to prove tightness of X_T. By the Mitoma theorem [14, Mitoma 1983] it is sufficient to show tightness of the real processes $\langle X_T, \phi\rangle$ for all $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$. This can be done using a criterion [1, Theorem 12.3]. Detailed examination of the proof in [4] reveals that only the covariance function of the N_t^{Poiss} is needed [4, Section 3.1]. One can see that the covariance function (3.10) is essentially the same as for the binary branching. Hence the proof from [4] still holds for the new family of processes.

3.1.2 The Laplace functional

The second step uses the space-time method. According to (1.10) we define $\tilde{X}_T$ (from now on $\tau = 1$). To establish the convergence we use Laplace functional.

By the Poisson initial condition we have (this equation is the same as [4, (3.10)]

$$\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T,\Phi\right\rangle\right\}=\exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,s)\,dsdx\right\}\exp\left\{\int_{\mathbb{R}^d}-v_T(x,0,T)\,dx\right\}, \tag{3.11}$$

Now we make similar computations to [4, (3.21)-(3.23)]. By combining (3.11) and (3.8) we obtain:

$$\begin{aligned}\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T,\Phi\right\rangle\right\}&=\exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,s)\,dsdx\right\}\\&\quad\cdot\exp\left\{-\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,T-s)\left(1-v_T(x,T-s,s)\right)-VG\left(v_T(x,T-s,s)\right)dsdx\right\}\\&=\exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,T-s)\,v_T(x,T-s,s)+VG\left(v_T(x,T-s,s)\right)dsdx\right\}\end{aligned}$$

The last expression can be rewritten as:

$$\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T,\Phi\right\rangle\right\}=\exp\left\{V\left(I_1(T)+I_2(T)\right)+I_3(T)\right\}, \tag{3.12}$$

where

$$\begin{aligned}I_1(T)&=\int_0^T\int_{\mathbb{R}^d}\frac{M}{2}\left(\int_0^s \mathcal{T}_u\Psi_T(\cdot,T+u-s)(x)\,du\right)^2dxds,\\I_2(T)&=\int_0^T\int_{\mathbb{R}^d}\left[G\left(v_T(x,T-s,s)\right)-\frac{M}{2}\left(\int_0^s \mathcal{T}_u\Psi_T(\cdot,T+u-s)(x)\,du\right)^2\right]dxds,\end{aligned} \tag{3.13}$$

$$I_3(T)=\int_0^T\int_{\mathbb{R}^d}\Psi_T(x,T-s)\,v_T(x,T-s,s)\,dxds.$$

To complete the proof we have to compute limits as $T\to+\infty$. We claim

$$I_1(T)\to\frac{MK}{2V}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,t)\,\Phi(y,s)\,dxdyC_h(s,t)\,dsdt, \tag{3.14}$$

$$I_2(T)\to 0,$$

$$I_3(T)\to 0,$$

Combining (3.12) with the above limits we obtain

$$\lim_{T\to+\infty}\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T,\Phi\right\rangle\right\}=\exp\left\{\frac{MK}{2}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,t)\,\Phi(y,s)\,dxdyC_h(s,t)\,dsdt\right\} \tag{3.15}$$

hence the limit process X_T is a Gaussian process with covariance (2.1).

3.1.3 Convergence proofs

$I_1(T)$ does not depend on F so it can be evaluated in the same way as in [4, (3.32)-(3.34)].
Let us now deal with $I_3(T)$. By using (3.9) we obtain

$$I_3(T) \le \int_0^T \int_{\mathbb{R}^d} \Psi_T(x, T-s) \int_0^s \mathcal{T}_u \Psi_T(\cdot, T-u)\, du dx ds \le \frac{C}{F_T^2} \int_0^T \int_{\mathbb{R}^d} \varphi(x) \int_0^s \mathcal{T}_u \varphi(x)\, du dx ds$$

Now the rest of the proof goes along the same lines as in [4].
We will turn to $I_2(T)$ which is a little more intricate. Combining (3.13) and property 4 from Fact 3.1

$$I_2(T) = \int_0^T \int_{\mathbb{R}^d} \left[\frac{M}{2} \left[v_T(\dots)^2 - \left(\int_0^s \mathcal{T}_u \Psi_T(\cdot, T+u-s)(x)\, du \right)^2 \right] + g(v_T(\dots))\, v_T(\dots)^2 \right] dx ds =$$

$$= \frac{M}{2} I_2'(T) + I_2''(T),$$

where

$$I_2'(T) = \int_0^T \int_{\mathbb{R}^d} v_T(x, T-s, s)^2 - \left(\int_0^s \mathcal{T}_u \Psi_T(\cdot, T+u-s)(x)\, du \right)^2 dx ds,$$

$$I_2''(T) = \int_0^T \int_{\mathbb{R}^d} g(v_T(x, T-s, s))\, v_T(x, T-s, s)^2\, dx ds \tag{3.16}$$

By inequality (3.9) we have

$$0 \le -I_2'(T) = \int_0^T \int_{\mathbb{R}^d} \left[(n_T(x, T-s, s))^2 - (v_T(x, T-s, s))^2 \right].$$

Combining (3.8) and (3.3) yields

$$0 \le n_T(x, T-s, s) - v_T(x, T-s, s) = \int_0^s \mathcal{T}_{s-u} \left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + VG(v_T(\cdot, T-u, u)) \right](x)\, du = (*).$$

We have $\mathcal{T}_s \Psi \ge 0$ for $\Psi \ge 0$ which is a direct consequence of the fact that $\mathcal{T}$ is the semigroup of a Markov process. By Fact 3.3 we have $c(F)$ such that $F(v) \le \frac{c(F)}{2} v^2$. Hence

$$\begin{aligned} (*) &\le \int_0^s \mathcal{T}_{s-u} \left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + c(F) \frac{V}{2} v_T(\cdot, T-u, u)^2 \right](x)\, du \\ &\le \max(1, c(F)) \int_0^s \mathcal{T}_{s-u} \left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + \frac{V}{2} v_T(\cdot, T-u, u)^2 \right](x)\, du \\ &\le \max(1, c(F)) \int_0^s \mathcal{T}_{s-u} \left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + \frac{V}{2} n_T(\cdot, T-u, u)^2 \right](x)\, du. \end{aligned}$$

Except of the constant $c(F)$ the last expression does not depend on F.
Next we consider

$$n_T(x,T-s,s)+v_T(x,T-s,s) \leq 2n_T(x,T-s,s) \leq 2\int_0^s \mathcal{T}_{s-u}\Psi(\cdot,T-u)(x)\,du.$$

The rest of the proof goes along the lines of the proof in [4, inequalities (3.39)-(3.42)] and hence we acquire $I_2'(T) \to 0$.
Before proving the convergence of $I_2''(T)$ we state two facts:

Fact 3.7. $n_T(x,T-s,s) \to 0$ *in uniformly* $x \in \mathbb{R}^d$, $s \in [0,T]$ *as* $T \to +\infty$.

Proof.

$$n_T(x,T-s,s) = \int_0^s \mathcal{T}_{s-u}\Psi_T(\cdot,T-u)\,du =$$
$$\frac{1}{F_T}\int_0^s \mathcal{T}_{s-u}\varphi(x)\,\chi\left(\frac{T-u}{T}\right)du \leq$$
$$\frac{C}{F_T}\int_0^{+\infty} \mathcal{T}_u\varphi(x)\,du = \frac{C_1}{F_T}\int_{\mathbb{R}^d}\frac{\varphi(y)}{|x-y|^{d-\alpha}}dy \leq \frac{C_2}{F_T} \to 0.$$

The last line contains the definition of the potential operator of the semigroup $\mathcal{T}_t$ which is bounded in respect to x (this can be found in [13, Lemma 5.3]). $\square$

Fact 3.8. *The following convergence holds:*

$$\int_0^T \int_{\mathbb{R}^d} v_T(x,T-s,s)^2 \to c'(\Psi) \ \ \textit{as}\ T \to +\infty.$$

Proof. One easily checks that

$$2\frac{I_1(T)}{M} + I_2'(T) = \int_0^T \int_{\mathbb{R}^d} v_T(x,T-s,s)^2.$$

Hence the result follows from (3.14) and $I_2'(T) \to 0$ as $T \to 0$. $\square$

It is now easy to prove the convergence of I_2''. From Fact 3.1 property 4 we know that for given $\epsilon > 0$ we can choose such δ that $\forall_{x\in(-\delta,\delta)}\,|g(x)| \leq \epsilon$. Fact 3.7 provides us with T_0 such that $\forall_{T\geq T_0}\, n_T(x,T-s,s) < \delta$. Combining this with (3.9) we obtain $\forall_{T\geq T_0}\, g(v_T(x,T-s,s)) \leq \epsilon$. Hence for $T > T_0$ holds:

$$|I_2''(T)| \leq \epsilon \int_0^T \int_{\mathbb{R}^d} v_T^2(x,T-s,s)\,dxds \to \epsilon c'(\Psi).$$

Since ϵ was chosen arbitrary we have convergence: $I_2''(T) \to 0$ hence also $I_2(T) \to 0$ as $T \to +\infty$.

Thus we obtained the limits for I_1, I_2 and I_3 and the proof of Theorem 2.1 is completed.

3.2 Proof of Theorem 2.2

3.2.1 Tightness

We begin by claiming that the family $\{X_T\}_{T>0}$ is tight. Close examination of [4, Section 3.1] reveals that only the covariance function of the underlying system is significant for the proof. By [3, (3.16)] we know that the covariance function of the branching system is of the same form as the covariance function of the non-branching system with the Poisson initial condition. From this we conclude that X_T is tight.

3.2.2 Laplace functional for $\tilde{X}_T$

We consider $\tilde{X}_T$ defined by (1.10). Using (1.3) and interchanging the order of integration we obtain

$$\left\langle \tilde{X}_T, \Phi \right\rangle = \frac{T}{F_T} \left[\int_0^1 \left\langle N_{Ts}, \Psi\left(\cdot, s\right) \right\rangle ds - \left\langle \lambda, \int_0^1 \Psi\left(\cdot, s\right) ds \right\rangle \right].$$

To prove the convergence of $\tilde{X}_T$ to $\tilde{X}$ we will use its Laplace functional

$$\begin{aligned} \mathbb{E} \exp\left\{ - \left\langle \tilde{X}_T, \Phi \right\rangle \right\} = \quad & \exp\left\{ \int_{\mathbb{R}^d} \int_0^T \Psi_T\left(x, t\right) dt dx \right\} \\ & \mathbb{E} \exp\left\{ - \int_0^T \left\langle N_s, \Psi_T\left(\cdot, s\right) \right\rangle ds \right\}, \end{aligned} \tag{3.17}$$

It is easy to check that

$$\mathbb{E}\left(\exp\left\{ - \int_0^T \left\langle N_s, \Psi_T\left(\cdot, s\right) \right\rangle ds \right\} | N_0 = \mu \right) = \exp\left\{ \left\langle \mu, \ln w_T \right\rangle \right\}, \tag{3.18}$$

where

$$w_T\left(x\right) = \mathbb{E} \exp\left\{ - \int_0^T \left\langle N_s^x, \Psi_T\left(\cdot, t\right) \right\rangle dt \right\}$$

Now we check that $0 \leq -\ln(w_T)$ is integrable. For T big enough by Fact 3.7 and inequity (3.9) we have $0 \leq v_T \leq c < 1$. Hence there exists a constant C such that we have $-\ln(w_T) = -\ln(1 - v_T) \leq C v_T \leq C n_T$. A trivial verifications shows that $n_T \in \mathcal{L}^1(\mathbb{R}^d)$ so by (1.1) and (3.18) we obtain

$$\mathbb{E} \exp\left\{ - \int_0^T \left\langle N_s, \Psi_T\left(\cdot, s\right) \right\rangle ds \right\} = \mathbb{E}\left(\mathbb{E}\left(\exp\left\{ - \int_0^T \left\langle N_s, \Psi_T\left(\cdot, s\right) \right\rangle ds \right\} | N_0 \right) \right) =$$

$$\exp\left\{ \left\langle \lambda, w_T - 1 \right\rangle + V \int_0^{+\infty} \left\langle \lambda, H\left(W_T\left(\cdot, s\right)\right) \right\rangle ds \right\},$$

where W_T satisfies the equation

$$W_T(x,l) = \mathcal{T}_l w_T(x) + V\int_0^l \mathcal{T}_{l-s} H(W_T(\cdot,s))(x)\,ds$$

It will be a bit easier to deal with $V_T(x,l) = 1 - W_T(x,l)$. The equations have the form (let us recall that G is defined by (3.1))

$$\mathbb{E}\exp\left\{-\int_0^T \langle N_s, \Psi_T(\cdot,s)\rangle\,ds\right\} = \exp\left\{\langle\lambda, -v_T\rangle + V\int_0^{+\infty}\langle\lambda, G(V_T(\cdot,s))\rangle\,ds\right\}, \tag{3.19}$$

and

$$V_T(x,l) = \mathcal{T}_l v_T(x) - V\int_0^l \mathcal{T}_{l-s} G(V_T(\cdot,s))(x)\,ds, \tag{3.20}$$

W_T is defined by (1.2) with $\varphi(x) = -\ln w_T(x)$ ($w_T \in [0,1]$ hence φ is positive). One can easily see that the definition implies that $W_T \in [0,1]$. Consequently $V_T \in [0,1]$ which together with Fact 3.2 yields $G(V_T) \geq 0$. Hence we obtain an inequality

$$V_T(x,l) \leq \mathcal{T}_l v_T(x), \ \forall_{x\in\mathbb{R}^d, l\geq 0}. \tag{3.21}$$

Combining (3.17) and (3.19) we obtain

$$\begin{aligned}\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T, \Phi\right\rangle\right\} &= \exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,t)\,dtdx\right\}\exp\left\{-\int_{\mathbb{R}^d} v_T(x)\,dx\right\}\\ &\quad \exp\left\{V\int_0^{+\infty}\int_{\mathbb{R}^d} G(V_T(x,t))\,dxdt\right\} = A(T)\cdot B(T),\end{aligned}$$

where

$$A(T) = \exp\left\{\int_{\mathbb{R}^d}\int_0^T \Psi_T(x,t)\,dtdx\right\}\exp\left\{-\int_{\mathbb{R}^d} v_T(x)\,dx\right\},$$

$$B(T) = \exp\left\{V\int_0^{+\infty}\int_{\mathbb{R}^d} G(V_T(x,t))\,dxdt\right\}.$$

Let us note that A is the same as (3.11) in the first proof hence we know that its limit is given by (3.15).

3.2.3 Limit of B

To complete the proof the limit $\lim_{T\to+\infty} B(T)$ has to be calculated. It suffices to consider

$$\int_0^{+\infty}\int_{\mathbb{R}^d} G(V_T(x,t))\,dxdt. \tag{3.22}$$

Using Fact 3.1, property 4, we split it in the following way

$$\int_0^{+\infty}\int_{\mathbb{R}^d} G\left(V_T\left(\cdot,t\right)\right)dxdt = \frac{M}{2}\left(B_1\left(T\right)+B_2\left(T\right)+B_3\left(T\right)\right)+B_4\left(T\right),$$

where

$$B_1\left(T\right)=\int_0^{+\infty}\int_{\mathbb{R}^d} V_T\left(x,t\right)^2-\left(\mathcal{T}_t v_T\left(x\right)\right)^2 dxdt,$$

$$B_2\left(T\right)=\int_0^{+\infty}\int_{\mathbb{R}^d} \left(\mathcal{T}_t v_T\left(x\right)\right)^2-\left(\mathcal{T}_t n_T\left(x\right)\right)^2 dxdt,$$

$$B_3\left(T\right)=\int_0^{+\infty}\int_{\mathbb{R}^d} \left(\mathcal{T}_t n_T\left(x\right)\right)^2 dxdt,$$

$$B_4\left(T\right)=\int_0^{+\infty}\int_{\mathbb{R}^d} g\left(V_T\left(x,t\right)\right)V_T\left(x,t\right)^2.$$

We will prove the following limits (let us recall that we assume (3.2) for simplicity)

$$B_1\left(T\right)\to 0,$$

$$B_2\left(T\right)\to 0,$$

$$B_3\left(T\right)\to \frac{K}{2V}\left\langle\lambda,\varphi\right\rangle^2\int_0^1\int_0^1\left\{-u_1^h-u_2^h+\left(u_1+u_2\right)^h\right\}\psi\left(u_1\right)\psi\left(u_2\right)du_1du_2,$$

$$B_4\left(T\right)\to 0,$$

as $T\to+\infty$.

Limit of B_1
By (3.21) we obtain

$$0\leq -B_1\left(T\right)=\int_0^{+\infty}\left(\int_{\mathbb{R}^d}\left(\mathcal{T}_t v_T\left(x\right)\right)^2-V_T\left(x,t\right)^2dx\right)dt=$$

$$\int_0^{+\infty}\int_{\mathbb{R}^d}\left(\mathcal{T}_t v_T\left(x\right)-V_T\left(x,t\right)\right)\left(\mathcal{T}_t v_T\left(x\right)V+_T\left(x,t\right)\right)dxdt\leq$$

Combining this with inequality (3.21) and equation (3.20) we have

$$\int_0^{+\infty}\int_{\mathbb{R}^d}\left(V\int_0^t\mathcal{T}_{t-t'}G\left(V_T\left(\cdot,t'\right)\right)\left(x\right)dt'\right)\left(2\mathcal{T}_t v_T\left(x\right)\right)dxdt=$$

Taking into account the form of G (Fact 3.1, property 4)

$$B_{11}\left(T\right)+B_{12}\left(T\right),$$

where

$$B_{11}\left(T\right)=\int_0^{+\infty}\int_{\mathbb{R}^d}\left(V\frac{M}{2}\int_0^t\mathcal{T}_{t-t'}V_T\left(\cdot,t'\right)^2\left(x\right)dt'\right)\left(2\mathcal{T}_t v_T\left(x\right)\right)dxdt,$$

$$B_{12}(T)=\int_0^{+\infty}\int_{\mathbb{R}^d}\left(V\int_0^t \mathcal{T}_{t-t'}g\left(V_T\left(\cdot,t'\right)\right)V_T\left(\cdot,t'\right)^2(x)\,dt'\right)\left(2\mathcal{T}_t v_T(x)\right)dxdt$$

Once again we use inequality (3.21)

$$B_{11}(T)\leq VM\int_0^{+\infty}\int_{\mathbb{R}^d}\left(\int_0^t \mathcal{T}_{t-t'}\left(\mathcal{T}_{t'}v_T(\cdot)\right)^2(x)\,dt'\right)\left(\mathcal{T}_t v_T(x)\right)dxdt=$$

$$VM\int_0^{+\infty}\int_0^t\int_{\mathbb{R}^d}\mathcal{T}_{t-t'}\left(\mathcal{T}_{t'}v_T(\cdot)\right)^2(x)\,\mathcal{T}_t v_T(x)\,dxdt'dt\leq$$

Applying (3.9) twice

$$VM\int_0^{+\infty}\int_0^t\int_{\mathbb{R}^d}\mathcal{T}_{t-t'}\left(\mathcal{T}_{t'}n_T(\cdot)\right)^2(x)\,\mathcal{T}_t n_T(x)\,dxdt'dt=$$

$$MV\int_0^{+\infty}\int_0^t\int_{\mathbb{R}^d}\mathcal{T}_{t'}n_T(x)\,\mathcal{T}_{t'}n_T(x)\,\mathcal{T}_{2t-t'}n_T(x)\,dxdt'dt=$$

We use the Plancherel formula and (1.4)

$$\frac{MV}{(2\pi)^{2d}}\int_0^{+\infty}\int_0^t\int_{\mathbb{R}^{2d}}\widehat{\mathcal{T}_{t'}n_T}(z_1)\,\widehat{\mathcal{T}_{t'}n_T}(z_2)\overline{\widehat{\mathcal{T}_{2t-t'}n_T}}(z_1+z_2)\,dz_1dz_2dt'dt=$$

$$\frac{MV}{(2\pi)^{2d}}\int_0^{+\infty}\int_0^t\int_{\mathbb{R}^{2d}}e^{-t'|z_1|^\alpha}\widehat{n}_T(z_1)\,e^{-t'|z_2|^\alpha}\widehat{n}_T(z_2)e^{-(2t-t')|z_1+z_2|^\alpha}\overline{\widehat{n}_T}(z_1+z_2)\,dz_1dz_2dt'dt=$$

$$\frac{MV}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\widehat{n}_T(z_1)\,\widehat{n}_T(z_2)\overline{\widehat{n}_T}(z_1+z_2)\int_0^{+\infty}\int_0^t e^{-t'|z_1|^\alpha}e^{-t'|z_2|^\alpha}e^{-(2t-t')|z_1+z_2|^\alpha}dt'dtdz_1dz_2=$$

$$\frac{MV}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\frac{1}{2\left|z_1+z_2\right|^\alpha\left(\left|z_1\right|^\alpha+\left|z_2\right|^\alpha+\left|z_1+z_2\right|^\alpha\right)}\widehat{n}_T(z_1)\,\widehat{n}_T(z_2)\overline{\widehat{n}_T}(z_1+z_2)\,dz_1dz_2=(*)$$

Before proceeding further we will estimate $\widehat{n}_T$

$$\left|\widehat{n}_T(z,r,t)\right|=\left|\widehat{\int_0^t\mathcal{T}_{t-s}\Psi_T(\cdot,r+t-s)\,ds}(z)\right|=$$

$$\left|\frac{1}{F_T}\int_0^t e^{-(t-s)|z|^\alpha}\widehat{\varphi}(z)\,\chi_T(r+t-s)\,ds\right|\leq$$

$$\frac{\sup\chi}{F_T}\left|\widehat{\varphi}(z)\right|\int_0^t e^{-(t-s)|z|^\alpha}ds\leq$$

Hence

$$\left|\widehat{n}_T(z,r,t)\right|\leq\frac{C}{F_T}\frac{\left|\widehat{\varphi}(z)\right|}{|z|^\alpha}\left[1-e^{-t|z|^\alpha}\right]\tag{3.23}$$

and this immediately implies (see (3.7))

$$\left|\widehat{n}_T(z)\right|\leq\frac{C}{F_T}\frac{1}{|z|^\alpha}\left[1-e^{-T|z|^\alpha}\right].\tag{3.24}$$

Here, and in what follows, C denotes a generic constant.
Coming back to $(*)$ and using the last inequality we obtain

$$|(*)| \le \frac{C}{F_T^3} \int_{\mathbb{R}^{2d}} \frac{1}{2\left|z_1+z_2\right|^{\alpha}\left(\left|z_1\right|^{\alpha}+\left|z_2\right|^{\alpha}+\left|z_1+z_2\right|^{\alpha}\right)} \frac{1}{\left|z_1\right|^{\alpha}}\left[1-e^{-T\left|z_1\right|^{\alpha}}\right]$$
$$\frac{1}{\left|z_2\right|^{\alpha}}\left[1-e^{-T\left|z_2\right|^{\alpha}}\right] \frac{1}{\left|z_1+z_2\right|^{\alpha}}\left[1-e^{-T\left|z_1+z_2\right|^{\alpha}}\right] dz_1 dz_2 \quad =$$

Substituting $T^{1/\alpha} z_1 = y_1$ and $T^{1/\alpha} z_2 = y_2$ yields

$$\frac{CT^5}{F_T^3 T^{2\frac{d}{\alpha}}} \int_{\mathbb{R}^{2d}} \frac{1}{\left|y_1+y_2\right|^{\alpha}\left(\left|y_1\right|^{\alpha}+\left|y_2\right|^{\alpha}+\left|y_1+y_2\right|^{\alpha}\right)} \frac{1}{\left|y_1\right|^{\alpha}}\left[1-e^{-\left|y_1\right|^{\alpha}}\right]$$
$$\frac{1}{\left|y_2\right|^{\alpha}}\left[1-e^{-\left|y_2\right|^{\alpha}}\right] \frac{1}{\left|y_1+y_2\right|^{\alpha}}\left[1-e^{-\left|y_1+y_2\right|^{\alpha}}\right] dy_1 dy_2 \quad \le$$
$$B_{11}'(T) \cdot B_{11}'',$$

where

$$B_{11}'(T) = \frac{C' T^5}{F_T^3 T^{2\frac{d}{\alpha}}}$$

$$B_{11}'' = \int_{\mathbb{R}^{2d}} \frac{1}{\left|y_1+y_2\right|^{\alpha}\left(\left|y_1\right|^{\alpha}+\left|y_2\right|^{\alpha}+\left|y_1+y_2\right|^{\alpha}\right)} \frac{1}{\left|y_1\right|^{\alpha}}\left[1-e^{-\left|y_1\right|^{\alpha}}\right]$$
$$\frac{1}{\left|y_2\right|^{\alpha}}\left[1-e^{-\left|y_2\right|^{\alpha}}\right] \frac{1}{\left|y_1+y_2\right|^{\alpha}}\left[1-e^{-\left|y_1+y_2\right|^{\alpha}}\right] dy_1 dy_2$$

The integral B_{11}'' is finite which will be proved in Fact 4.1. The expression $B_{11}'(T)$ can be evaluated

$$B_{11}'(T) = T^{\frac{10-3\left(3-\frac{d}{\alpha}\right)-4\frac{d}{\alpha}}{2}} = T^{\frac{1-\frac{d}{\alpha}}{2}}$$

and as $1-\frac{d}{\alpha}<0$ $B_{11}'(T) \to 0$ hence the convergence: $B_{11}(T) \to 0$ is obtained too.
From Fact 3.7 and inequalities (3.9) and (3.21) we know $V_T(x,l) \to 0$ uniformly as $T \to 0$ and so $g\left(V_T(x,l)\right) \le \epsilon$ for T sufficiently large hence

$$B_{12}(T) \le \epsilon \int_0^{+\infty} \int_{\mathbb{R}^d} \left(V \int_0^t \mathcal{T}_{t-t'} V_T(\cdot,t')^2(x)\, dt'\right)\left(2\mathcal{T}_t v_T(x)\right) dx dt \le \frac{2\epsilon}{M} B_{11}(T)$$

thus $B_{12}(T) \to 0$ and $B_1(T) \to 0$ too.

Limit of B_2
Let us first estimate expression $n_T - v_T$ using (3.8) and (3.3)

$$\begin{aligned} n_T(x) - v_T(x) = & \int_0^T \mathcal{T}_{T-u} \Psi_T(\cdot, T-u)(x)\, du \\ & - \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\left(1-v_T(\cdot, T-u, u)\right) - VG\left(v_T(\cdot, T-u, u)\right)\right](x)\, du \end{aligned}$$

$$n_T(x) - v_T(x) = \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + VG\left(v_T(\cdot, T-u, u)\right)\right](x)\, du \leq$$

Applying Fact 3.3

$$\int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + Vc\left(v_T(\cdot, T-u, u)\right)^2\right](x)\, du,$$

where c is a constant. By inequality (3.9)

$$n_T(x) - v_T(x) \leq \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2\right](x)\, du \tag{3.25}$$

We have

$$0 \leq -B_2(T) = \int_0^{+\infty}\left(\int_{\mathbb{R}^d}\left(\mathcal{T}_t n_T(x)\right)^2 - \left(\mathcal{T}_t v_T(x)\right)^2 dx\right) dt =$$

$$\int_0^{+\infty}\int_{\mathbb{R}^d}\left(\mathcal{T}_t\left(n_T(\cdot) - v_T(\cdot)\right)(x)\right)\left(\mathcal{T}_t\left(v_T(\cdot) + n_T(\cdot)\right)(x)\right) dx dt \leq$$

Applying (3.9) and (3.25)

$$2\int_0^{+\infty}\int_{\mathbb{R}^d}\mathcal{T}_t\left\{\int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2\right] du\right\}(x)$$
$$\mathcal{T}_t n_T(x)\, dx dt =$$

Now we apply the Plancherel formula

$$\frac{2}{(2\pi)^d}\int_0^{+\infty}\int_{\mathbb{R}^d} e^{-2t|z|^\alpha}\int_0^T \widehat{\mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2(\cdot)\right]}(z)\, du$$
$$\widehat{n}_T(z)\, dz dt \quad =$$

Interchanging the order of integration and integrating with respect to t we get

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\int_0^T \widehat{\mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2(\cdot)\right]}(z)\, du$$
$$\widehat{n}_T(z)\, dz = c'\left(B_{21}(T) + B_{22}(T)\right),$$

where

$$B_{21}(T) = \int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left\{\int_0^T \widehat{\mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u)\right]}(z)\, du\right\}\widehat{n}_T(z)\, dz,$$

$$B_{22}(T) = \int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left\{\int_0^T \widehat{\mathcal{T}_{T-u}\left[Vc\left(n_T(\cdot, T-u, u)\right)^2(\cdot)\right]}(z)\, du\right\}\widehat{n}_T(z)\, dz.$$

We shall compute $\lim_{T\to+\infty} B_{21}(T)$ first. We have

$$B_{21}(T) = \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \left\{ \int_0^T e^{-(T-u)|z|^\alpha} \Psi_T \widehat{(\cdot, T-} u) * n_T \widehat{(\cdot, T-u}, u)(z) \right\} \widehat{n}_T(z) dz.$$

The inner convolution can be estimated using inequality (3.23) and simplification (3.2)

$$\left|\Psi_T \widehat{(\cdot, T-} u) * n_T \widehat{(\cdot, T-} u, u)(z)\right| = \left|\chi_T (T-u) \widehat{\varphi_T} (\cdot) * n_T \widehat{(\cdot, T-u}, u)(z)\right| =$$

$$\left|\chi_T (T-u) \int_{\mathbb{R}^d} \widehat{\varphi}_T (z-x) \widehat{n}_T (x, T-u, u) dx\right| \leq$$

$$\frac{c(\chi)}{F_T^2} \chi_T (T-u) \int_{\mathbb{R}^d} |\widehat{\varphi}(z-x) \widehat{\varphi}(x)| \frac{1}{|x|^\alpha} dx \leq \frac{C}{F_T^2}$$

In the last inequality we use the fact that $\hat{\varphi}$ is bounded and $\frac{\widehat{\varphi}(x)}{|x|^\alpha}$ is integrable. Hence we have inequality

$$\left|\Psi_T \widehat{(\cdot, T-} u) * n_T \widehat{(\cdot, T-} u, u)(z)\right| \leq \frac{C}{F_T^2} \tag{3.26}$$

Thus B_{21} satisfies

$$|B_{21}(T)| \leq \frac{C}{F_T^2} \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \int_0^T e^{-(T-u)|z|^\alpha} du \cdot \widehat{n}_T(z) dz \leq$$

Using inequality (3.24) and integrating with respect to u

$$C' \frac{1}{F_T^3} \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha} \left[1 - e^{-T|z|^\alpha}\right] \frac{1}{|z|^\alpha} \left[1 - e^{-T|z|^\alpha}\right] dz =$$

Substituting $zT^{1/\alpha} = y$

$$C' \frac{T^3}{F_T^3 T^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} \frac{1}{|y|^\alpha} \frac{1}{|y|^\alpha} \left[1 - e^{-|y|^\alpha}\right] \frac{1}{|y|^\alpha} \left[1 - e^{-|y|^\alpha}\right] dy \leq B'_{21}(T) \cdot B''_{21},$$

where

$$B'_{21}(T) = C'' \frac{T^3}{F_T^3 T^{\frac{d}{\alpha}}}$$

$$B''_{21} = \int_{\mathbb{R}^d} \frac{1}{|y|^\alpha} \frac{1}{|y|^\alpha} \left[1 - e^{-|y|^\alpha}\right] \frac{1}{|y|^\alpha} \left[1 - e^{-|y|^\alpha}\right] dy$$

Is is clear that integral B''_{21} in the last expression is finite since in a neighborhood of 0 the integrated expression is proportional to $\frac{1}{|y|^\alpha}$ and it is $O\left(\frac{1}{|y|^{3\alpha}}\right)$ as $|y| \to +\infty$ (recall that $\alpha < d < 2\alpha$). Now only B'_{21} needs to be evaluated

$$B'_{21}(T) = C'' T^{\frac{6-3\left(3-\frac{d}{\alpha}\right)-2\frac{d}{\alpha}}{2}} = C'' T^{\frac{-3+\frac{d}{\alpha}}{2}}.$$

Hence it is obvious that $B'_{21}(T) \to 0$ as $T \to 0$ and so $\lim_{T\to 0} B_{21}(T) = 0$.
Before proceeding to B_{22} we will make the following estimation using inequality (3.23)

$$\left|\widehat{(n_T(\cdot, T-u, u))^2}\right|(z) = \left|\int_{\mathbb{R}^d} \widehat{n}_T(x, T-u, u)\, \widehat{n}_T(z-x, T-u, u)\, dx\right| \leq$$

$$\frac{C}{F_T^2} \int_{\mathbb{R}^d} \frac{1}{|x|^\alpha}\left[1-e^{-u|x|^\alpha}\right] \frac{1}{|z-x|^\alpha}\left[1-e^{-u|z-x|^\alpha}\right] dx \leq$$

Substitution $xu^{1/\alpha} = y$ yields

$$u^{2-\frac{d}{\alpha}} \frac{C}{F_T^2} \int_{\mathbb{R}^d} \frac{1}{|y|^\alpha}\left[1-e^{-|y|^\alpha}\right] \frac{1}{\left|zu^{1/\alpha}-y\right|^\alpha}\left[1-e^{-\left|zu^{1/\alpha}-y\right|^\alpha}\right] dy \leq \frac{C'}{F_T^2} u^{2-\frac{d}{\alpha}}$$

since the integral can be regarded as a convolution of $\mathcal{L}^2$ functions so it is bounded. This clearly implies

$$|B_{22}(T)| \leq \frac{C'}{F_T^2} \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \int_0^T e^{-(T-u)|z|^\alpha} u^{2-\frac{d}{\alpha}} du \cdot |\widehat{n}_T(z)|\, dz \leq$$

$$C' \frac{T^{2-\frac{d}{\alpha}}}{F_T^2} \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \int_0^T e^{-(T-u)|z|^\alpha} du \cdot |\widehat{n}_T(z)|\, dz \leq$$

Using inequality (3.24) we obtain

$$C'' \frac{T^{2-\frac{d}{\alpha}}}{F_T^3} \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha}\left(1-e^{-T|z|^\alpha}\right) \frac{1}{|z|^\alpha}\left(1-e^{-T|z|^\alpha}\right) dz =$$

Substituting $zT^{1/\alpha} = y$ we can rewrite the last expression as

$$C'' \frac{T^{5-\frac{d}{\alpha}}}{F_T^3 T^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} \frac{1}{|y|^\alpha} \frac{1}{|y|^\alpha}\left(1-e^{-|y|^\alpha}\right) \frac{1}{|y|^\alpha}\left(1-e^{-|y|^\alpha}\right) dy.$$

The integral is finite (the same proof as for B''_{21}) and

$$\frac{T^{5-\frac{d}{\alpha}}}{F_T^3 T^{\frac{d}{\alpha}}} = T^{\frac{10-2\frac{d}{\alpha}-3\left(3-\frac{d}{\alpha}\right)-2\frac{d}{\alpha}}{2}} = T^{\frac{1-\frac{d}{\alpha}}{2}}$$

which yields $B_{22}(T) \to 0$ as $T \to +\infty$

Limit of B_3
Applying the Plancherel formula to $B_3(T)$ we get

$$B_3(T) = \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^d} e^{-2t|z|^\alpha} \left(\widehat{n}_T(z)\right)^2 dz dt =$$

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\widehat{n}_T(z)\right)^2 \int_0^\infty e^{-2t|z|^\alpha} dt dz =$$

$$\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left(\widehat{n}_T(z)\right)^2 dz =$$

$$\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left(\int_0^T e^{-(T-u)|z|^\alpha}\widehat{\varphi}_T(z)\,\chi_T(T-u)\,du\right)^2 dz =$$

$$\frac{1}{2(2\pi)^d}\frac{1}{F_T^2}\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left(\int_0^T e^{-u|z|^\alpha}\widehat{\varphi}(z)\,\chi_T(u)\,du\right)^2 dz =$$

Substituting $u' = u/T$

$$\frac{1}{2(2\pi)^d}\frac{T^2}{F_T^2}\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}\left(\int_0^1 e^{-Tu'|z|^\alpha}\widehat{\varphi}(z)\,\chi(u')\,du'\right)^2 dz =$$

$$\frac{1}{2(2\pi)^d}\frac{T^2}{F_T^2}\int_0^1\int_0^1\int_{\mathbb{R}^d}\frac{1}{|z|^\alpha}e^{-T(u_1+u_2)|z|^\alpha}\left(\widehat{\varphi}(z)\right)^2\chi(u_1)\,\chi(u_2)\,du_1du_2dz =$$

Let $z = [T(u_1+u_2)]^{-\frac{1}{\alpha}}\,y$

$$\frac{1}{2(2\pi)^d}\frac{T^{3-\frac{d}{\alpha}}}{F_T^2}\int_0^1\int_0^1\int_{\mathbb{R}^d}(u_1+u_2)\,\frac{1}{|y|^\alpha}e^{-|y|^\alpha}\left(\widehat{\varphi}\left([T(u_1+u_2)]^{-\frac{1}{\alpha}}\,y\right)\right)^2 (u_1+u_2)^{-\frac{d}{\alpha}}\,\chi(u_1)\,\chi(u_2)\,du_1du_2dy$$

Therefore by Lebesgue's dominated convergence theorem we obtain the limit of $B_3(T)$

$$\lim_{T\to+\infty}B_3(T) = \frac{1}{2(2\pi)^d}\int_0^1\int_0^1\int_{\mathbb{R}^d}(u_1+u_2)^{1-\frac{d}{\alpha}}\,\frac{1}{|y|^\alpha}e^{-|y|^\alpha}\left(\widehat{\varphi}(0)\right)^2\chi(u_1)\,\chi(u_2)\,du_1du_2dy =$$

$$\frac{\Gamma\left(\frac{d}{\alpha}-1\right)}{2^d\alpha\Gamma\left(\frac{d}{2}\right)\pi^{\frac{d}{2}}}\langle\lambda,\varphi\rangle^2\int_0^1\int_0^1(u_1+u_2)^{1-\frac{d}{\alpha}}\,\chi(u_1)\,\chi(u_2)\,du_1du_2 =$$

Integrating by parts

$$\frac{K}{2V}\langle\lambda,\varphi\rangle^2\int_0^1\int_0^1\left\{-u_1^h-u_2^h+(u_1+u_2)^h\right\}\psi(u_1)\,\psi(u_2)\,du_1du_2$$

Limit of B_4
Firstly, let us notice that

$$B_1(T)+B_2(T)+B_3(T) = \int_0^{+\infty}\int_{\mathbb{R}^d}V_T(x,t)^2\,,$$

and hence

$$\int_0^{+\infty}\int_{\mathbb{R}^d}V_T(x,t)^2 \to_{T\to+\infty} C.$$

Secondly by Fact 3.7 and inequalities (3.21) and (3.9) we know $V_T(x) \to 0$ uniformly as $T \to 0$. Hence $g(W_T(x)) \leq \epsilon$ for T sufficiently large so

$$|B_4(T)| \leq \epsilon \int_0^{+\infty} \int_{\mathbb{R}^d} V_T(x,t)^2,$$

which clearly implies that $B_4(T) \to 0$ as $T \to +\infty$.

Putting the results together
Combining the previous results we conclude

$$\lim_{T\to+\infty} B(T) = \exp\left\{ \frac{MK}{4} \langle \lambda, \varphi \rangle^2 \int_0^1 \int_0^1 \left\{ -u_1^h - u_2^h + (u_1+u_2)^h \right\} \psi(u_1)\psi(u_2)\, du_1 du_2 \right\}$$

And finally by (3.15)

$$\lim_{T\to+\infty} A(T)\, B(T) = \exp\left\{ \frac{MK}{2} \langle \lambda, \varphi \rangle^2 \int_0^1 \int_0^1 c_h(u_1,u_2)\, \psi(u_1)\psi(u_2)\, du_1 du_2 \right\},$$

where c_h is the covariance function of fractional Brownian motion defined by (1.9). This Laplace functional defines a process $\tilde{X}_T$ corresponding to the Gaussian process X_T with the covariance (2.2) hence Theorem 2.2 is proved.

4 Appendix

The appendix contains a technical fact used in the main proof.

Fact 4.1.

$$\int_{\mathbb{R}^{2d}} \frac{1}{|y_1+y_2|^\alpha \left(|y_1|^\alpha + |y_2|^\alpha + |y_1+y_2|^\alpha\right)} \frac{1}{|y_1|^\alpha}\left[1 - e^{-|y_1|^\alpha}\right]$$
$$\frac{1}{|y_2|^\alpha}\left[1 - e^{-|y_2|^\alpha}\right] \frac{1}{|y_1+y_2|^\alpha}\left[1 - e^{-|y_1+y_2|^\alpha}\right] dy_1 dy_2 < +\infty$$

Proof. Substituting $x = y_1 + y_2$ and $z = y_2$ we get

$$\int_{\mathbb{R}^{2d}} \frac{1}{|x|^\alpha \left(|x|^\alpha + |z|^\alpha + |x-z|^\alpha\right)} \frac{1}{|x-z|^\alpha}\left[1 - e^{-|x-z|^\alpha}\right] \frac{1}{|z|^\alpha}\left[1 - e^{-|z|^\alpha}\right] \frac{1}{|x|^\alpha}\left[1 - e^{-|x|^\alpha}\right] dxdz =$$

$$\int_{\mathbb{R}^{2d}} \frac{1}{|x|^\alpha} \frac{1}{|x|^\alpha}\left[1 - e^{-|x|^\alpha}\right] \int_{\mathbb{R}^d} \frac{1}{|x|^\alpha + |z|^\alpha + |x-z|^\alpha} \frac{1}{|x-z|^\alpha}\left[1 - e^{-|x-z|^\alpha}\right] \frac{1}{|z|^\alpha}\left[1 - e^{-|z|^\alpha}\right] dzdx = (*)$$

Let us investigate now

$$\int_{\mathbb{R}^d} \frac{1}{|x|^\alpha + |z|^\alpha + |x-z|^\alpha} \frac{1}{|x-z|^\alpha}\left[1 - e^{-|x-z|^\alpha}\right] \frac{1}{|z|^\alpha}\left[1 - e^{-|z|^\alpha}\right] dz \leq$$

$$\int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha}\left[1 - e^{-|z|^\alpha}\right] \frac{1}{|x-z|^\alpha}\left[1 - e^{-|x-z|^\alpha}\right] dz \leq c \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha}\left[1 - e^{-|z|^\alpha}\right] dz$$

The last integral is finite since in the neighborhood of 0 the integrated function is $O\left(\frac{1}{|z|^\alpha}\right)$ and for big $|z|$ is $O\left(\frac{1}{|z|^{2\alpha}}\right)$. Going back to $(*)$ we obtain

$$(*) \leq c_2 \int_{\mathbb{R}^d} \frac{1}{|x|^\alpha} \frac{1}{|x|^\alpha} \left[1 - e^{-|x|^\alpha}\right] < c_3,$$

by the same reason as above. □

Acknowledgement. The author would like to thank his supervisor - prof. Tomasz Bojdecki - for much appreciated help given in general introduction to the branching systems theory and in writing this paper. The author wishes to thank also prof. Luis Gorostiza for several helpful comments.

References

[1] P. Billingsley, Convergence of Probability Measures., John Wiley&Sons, New York, 1968.

[2] T. Bojdecki, L.G. Gorostiza and S. Ramaswami, Convergence of $\mathcal{S}'$-valued processes and space time random fields, J. Funct. Anal. 66 (1986), pp. 21-41.

[3] T. Bojdecki, L.G. Gorostiza and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, Statist. Probab. Lett. 69 (2004), pp. 405-419.

[4] T. Bojdecki, L.G. Gorostiza and A. Talarczyk, Limit theorems for occupation time fluctuations of branching systems I: Long-range dependence, Stoch. Proc. Appl. 116 (2006), pp. 1-18.

[5] T. Bojdecki, L.G. Gorostiza and A. Talarczyk, Limit theorems for occupation time fluctuations of branching systems II: Critical and large dimensions Functional, Stoch. Proc. Appl. 116 (2006), pp. 19-35.

[6] T. Bojdecki, L.G. Gorostiza and A. Talarczyk, A long range dependence stable process and an infinite variance branching system, www.arxiv.org, math.PR/0511739 (2005).

[7] T. Bojdecki, L.G. Gorostiza and A. Talarczyk, Occupation time fluctuations of an infinite variance branching systems in large dimensions, www.arxiv.org, math.PR/0511745 (2005).

[8] M. Birkner and I. Zähle, Functional central limit theorems for the occupation time of the origin for branching random walks in $d \geq 3$, Weierstraß Insitut für Angewandte Analysis und Stochastik, Berlin, preprint No. 1011 (2005).

[9] J.D. Deuschel and K. Wang, Large deviations for the occupation time of a Poisson system of independent Brownian particles, Stoch. Proc. Appl. 52 (1994), pp. 183-209.

[10] L.G. Gorositza and E.R. Rodrigues, A stochastic model for transport of particulate matter in air: an asymptotic analysis. Acta Appl. Math. 59 (1999), pp. 21-43.

[11] L.G. Gorostiza and A. Wakolbinger, Long time behavior of critical branching particle systems and its applications, CRM Proc. and Lect. Notes Vol. 5 (1994), pp. 119-137.

[12] L.G. Gorostiza and A. Wakolbinger, Persistence criteria for a class of critical branching particle systems in continuous time. Ann. Probab. 19 (1991), pp. 266-288.

[13] I. Iscoe, A weighted occupation time for a class of measure-valued branching processes, Probab. Th. Rel. Fields 71 (1986), pp. 85-116.

[14] I. Mitoma, Tightness of probabilities on $C([0,1], \mathcal{S}')$ and $D([0,1], \mathcal{S}')$, Ann. Probab. 11 (1983), pp. 989-999.